Final Cheat Sheet

Useful Random Stuffs

Relationship Between Variables, Equations, and Solutions

For m equations and n variables. The coefficient matrix is $\mathbf{A}_{m\times n}$.

Configuration	$r = \text{rank}(\mathbf{A})$	Consistency	Possible Solution Types
$m < n$	$r = m$	Consistent	Infinitely many solutions ($n - r$ parameters)
	$r < m$	Inconsistent	No solutions
$m = n$	$r = n$	Consistent	Unique solution
	$r < n$	Consistent	Infinitely many solutions ($n - r$ parameters)
	$r < m$	Inconsistent	No solutions
$m > n$	$r = n$	Consistent	No solution or unique solution (depends on consistency)
	$r < n$	Consistent	Infinitely many solutions ($n - r$ parameters)
	$r < m$	Inconsistent	No solutions

Nilpotent Matrix

A matrix is nilpotent if $\mathbf{A}^k = \mathbf{0}$ where $k \leq n$. If $\mathbf{A}$ is nilpotent, $\mathbf{I} - \mathbf{A}$ is invertible.

Determinants

1. Triangular matrices have their determinants as the product of the diagonal entries.
2. cR_i multiplies the determinant by c.
3. $R_i \leftrightarrow R_j$ negates the determinant.

Check for Basis

Show that S is a basis for V, knowing that $|S| = \dim(V)$.

1. Show that $S \subseteq V$ and S is linearly independent.
2. OR Show that $V \subseteq \text{span}(S)$.

Note that dimension of subspace is the number of basis.

Orthogonal Basis

If $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is an orthogonal basis for a subspace V, then for any $\mathbf{v} \in V$:

$$\mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\right)\mathbf{u}_1 + \left(\frac{\mathbf{v} \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2}\right)\mathbf{u}_2 + \cdots + \left(\frac{\mathbf{v} \cdot \mathbf{u}_k}{\|\mathbf{u}_k\|^2}\right)\mathbf{u}_k.$$

If S is orthonormal, the denominators simplify to 1.

Linear Transformation and Basis

- Let S be the set of any basis. We can also write $T(\mathbf{v}) = [T]_S[\mathbf{v}]_S$
- If $\mathbf{P}$ is a transition matrix from standard basis to S, then $\mathbf{A} = [T]_S\mathbf{P}$.

Useful Theorems

Full Rank Equal Number of Columns

1. $\mathbf{A}_{m\times n}$ is full rank where $\text{rank}(\mathbf{A}) = n$.
2. $\text{Row}(\mathbf{A}) = \mathbb{R}^n$.
3. Columns of $\mathbf{A}$ are linearly independent.
4. $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution.
5. $\mathbf{A}^T\mathbf{A}$ is invertible with an order n.
6. $\mathbf{A}$ has left inverse.
7. $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $\mathbf{A}$ is injective if $n \leq m$.

Full Rank Equal Number of Rows

1. $\mathbf{A}_{m\times n}$ is full rank where $\text{rank}(\mathbf{A}) = m$.
2. $\text{Col}(\mathbf{A}) = \mathbb{R}^m$.
3. Rows of $\mathbf{A}$ are linearly independent.
4. $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$
5. $\mathbf{AA}^T$ is invertible with an order m.
6. $\mathbf{A}$ has right inverse.
7. $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $\mathbf{A}$ is surjective if $n \geq m$.

Equivalent Statements of Invertibility

1. $\mathbf{A}$ and $\mathbf{A}^T$ is invertible, that is $\det(\mathbf{A}) \neq 0$.
2. $\mathbf{A}$ is reduced to $\mathbf{I}_n$ in its RREF form, and could be represented as product of $\mathbf{E}$.
3. $\mathbf{Ax} = \mathbf{b}$ has a unique solution (trivial for homogeneous system).
4. The rows/columns are linearly independent and $\text{Row}(\mathbf{A}) = \text{Col}(\mathbf{A}) = \mathbb{R}^n$.
5. $\mathbf{A}$ is full rank where $\text{rank}(\mathbf{A}) = n$ and $\text{nullity}(\mathbf{A}) = 0$.
6. 0 is not an eigenvalue of $\mathbf{A}$.
7. Any linear transformations defined by $\mathbf{A}$ is bijective (both injective and surjective).

Rank-Nullity Theorem

$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = n.$

✓ Success

1. $\text{Row}(\mathbf{A})$ is the span of non-zero rows of RREF of $\mathbf{A}$.
2. $\text{Col}(\mathbf{A})$ is the span of pivot columns of $\mathbf{A}$ as determined from its RREF. It is also the range of linear transformation defined by $\mathbf{A}$.
3. $\text{Null}(\mathbf{A})$ is the solution space to $\mathbf{A}\mathbf{v} = \mathbf{0}$ and $\text{nullity}(\mathbf{A}) = \dim(\text{Null}(\mathbf{A}))$. It is also the kernel of linear transformation defined by $\mathbf{A}$.

Equivalent Conditions for Orthogonality (Square Matrix)

1. $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}$.
2. The columns of $\mathbf{A}$ form an orthonormal basis for $\mathbb{R}^n$.
3. The rows of $\mathbf{A}$ form an orthonormal basis for $\mathbb{R}^n$.

Equivalent Statements of Diagonalizability (Square Matrix)

1. There exists a set of eigenvectors of $\mathbf{A}$ that form a basis for $\mathbb{R}^n$.
2. The characteristic polynomials of $\mathbf{A}$ could be split into linear factors.
3. Strictly, $\dim(E_{\lambda_i}) = r_{\lambda_i}$.

Matrix Factorization

LU Factorization (Square Matrices)

1. Reduce $\mathbf{A}_n$ to upper triangular matrix $\mathbf{U}$ using $R_i + c_{ij}R_j$ from top to bottom ($i > j$).
2. Construct $\mathbf{L}$ from $\mathbf{I}_n$ by replacing zeros at entry (i, j) with $-c_{ij}$.
3. Write $\mathbf{A} = \mathbf{L}\mathbf{U}$.
4. If need to solve for $\mathbf{A}\mathbf{x} = \mathbf{b}$, solve for $\mathbf{L}\mathbf{y} = \mathbf{b}$ then solve for $\mathbf{U}\mathbf{x} = \mathbf{y}$.

QR Factorization

1. Perform Gram-Schmidt Orthogonalization on columns of $\mathbf{A}$ to get $\mathbf{Q}$.
2. Compute $\mathbf{R} = \mathbf{Q}^T\mathbf{A}$ where $\mathbf{R}$ is a upper-triangular matrix with positive diagonal entries.
3. Write $\mathbf{A} = \mathbf{Q}\mathbf{R}$.
4. To solve least square problems for $\mathbf{A}\mathbf{x} \approx \mathbf{b}$, solve $\mathbf{R}\mathbf{x} = \mathbf{Q}^T\mathbf{b}$ instead of $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$.

Note

Gram-Schmidt Orthogonalization

1. Set $\mathbf{v_1} = \mathbf{u_1}$.
2. Compute $\mathbf{v_n} = \mathbf{u_n} - \sum_{i=1}^{n} \left(\frac{\mathbf{v}_i \cdot \mathbf{u}_n}{\|\mathbf{v}_i\|^2} \mathbf{v}_i \right)$.
3. Normalize each $\mathbf{v}_i$ if required to obtain orthonormal set.

Diagonalization

1. Find eigenvalues of $\mathbf{A}_{n\times n}$.
2. For each eigenvalue, compute the corresponding eigenspace (null space of $\lambda\mathbf{I} - \mathbf{A}$).
3. Construct a matrix $\mathbf{P}$ from basis of each eigenspace.
4. Construct a matrix $\mathbf{D}$ by replacing diagonal entries in $\mathbf{I}_n$ with λ.
5. Write $\mathbf{A} = \mathbf{PDP}^{-1}$.

Tip

If $\mathbf{A} = \mathbf{A}^T$ (symmetric), we have $\mathbf{P}^{-1} = \mathbf{P}^T$. This is called orthogonal diagonalization. We may need to apply Gram-Schmidt Process vectors in $\mathbf{P}$ to get orthonormal basis.

SVD Factorization

1. Compute $\mathbf{A}^T\mathbf{A}$ and use this matrix for all following steps.
2. Find eigenvalues (sorted in descending order) and create a matrix $\mathbf{\Sigma}_{m\times n}$ by padding diagonal matrix of order r with 0. The diagonal matrix has $\sigma_i = \sqrt{\lambda_i}$ as their diagonal entries.
3. Construct a matrix $\mathbf{V}_{n\times n}$ using the same method for finding $\mathbf{P}$.
4. Construct a matrix $\mathbf{U}_{m\times m}$ from $\mathbf{u}_i = \mathbf{A}\mathbf{v}_i/\sigma_i$ (with normalization). If $r < m$, get the extra vectors by solving $(\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_r)^T\mathbf{x} = 0$.
5. Apply Gram-Schmidt Process on $\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_m\}$.
6. Write $\mathbf{A} = \mathbf{U\Sigma V}^T$.